Magnetic translation groups in n dimensions*

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Abstract

Magnetic translation groups are considered as central extensions of the translation group $T \simeq \mathbb{Z}^n$ by the group of factors (a gauge group) $\mathsf{U}(1)$. The obtained general formulae allow to consider a magnetic field as an antisymmetric tensor (of rank 2) and factor systems are determined by a transvection of this tensor with a tensor product $\mathbf{t} \otimes \mathbf{t}'$.

1 Introduction

The behaviour of electrons in crystalline (periodic) potentials in the presence of a constant (external) magnetic field has been studied since the thirties in many papers, amongst which works of Landau [1], Peierls [2], Onsager [3], Harper [4], and Azbel [5] seem to be ones of the most important. In the sixties Brown [6] and Zak [7, 8] (see also [9]) independently introduced and investigated the so-called *magnetic translation groups* (MTG).[1] Their results have been lately applied to a problem of the quantum Hall effect [11, 12] and relations with the Weyl–Heisenberg group (WHG) have been also studied [13]. Some interesting results have been presented lately by Geyler and Popov [14].

From the group-theoretical point of view magnetic translations can be considered as a projective (ray) representation of the translation group T of a crystal lattice (this is Brown's approach). However, projective representations of any group can be found as vector representations of its covering group, which can be constructed as a central extension of a given group by the group of factors being, in general, a subgroup of $\mathbb{C}^*$ (see, e.g., [15, 16]). This construction is a basis of Zak's considerations and it is also used in this paper, since obtained results allow us to see more general properties of MTG and we can get a deep insight in its (algebraic) structure.

The aim of this work is to investigate MTG for a n-dimensional crystal lattice, i.e. to investigate central extensions of $\mathbb{Z}^n$ by $\mathsf{U}(1)$. Such approach leads to interpretation of a magnetic field as an antisymmetric n-dimensional tensor of rank 2 and to determine mathematical background of flux quantization. All nonequivalent central extensions of $\mathbb{Z}^n$ by $\mathsf{U}(1)$ are found by the Mac Lane method for determination of the second cohomology group [17, 18] (see also [19] and references quoted therein), which becomes significantly simplified in the considered case of a central extension of two abellian groups. The important (especially from physical point of view) problem of determination of irreducible representations can be solved using the induction procedure (see, e.g., [16]) but it is left over to further considerations. Nevertheless, the obtained results can be compared with those of Brown and Zak and, therefore, one can find physical meaning of introduced parameters.

The facts indicating that MTG can be considered as a central extension are presented in Sec. 2. The relations with WHG are also pointed out. In Sec. 3 all non-equivalent central extensions of $\mathbb{Z}^n$ by $\mathsf{U}(1)$ are determined. Moreover, a labelling scheme for obtained extensions is introduced and its physical relevance is indicated. Some crucial, but cumbersome, calculations can be found in the appendix. As an example the case $n=3$ is considered in Sec. 3.9 and the results is compared with Brown's and Zak's ones.

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[1]The term *'magnetic translation group'* the first time was used by Peterson [10].

2 MTG, WHG and Central Extensions

The Weyl–Heisenberg group is generated by unitary operators (a and b are real constants)

$$\mathcal{Q}(a) = \exp(-\mathrm{i}aQ), \qquad \mathcal{P}(b) = \exp(-\mathrm{i}bP), \tag{1}$$

where (Q, P) is a pair of complementary (hermitian) operators, i.e.

$$[P, Q] = -\mathrm{i}\hbar, \tag{2}$$

The operators $\mathcal{Q}$ and $\mathcal{P}$ satisfy the following relation [20, 21] (see also [22], where a finite phase plane is discussed)

$$\mathrm{e}^{-\mathrm{i}bP}\mathrm{e}^{-\mathrm{i}aQ} = \mathrm{e}^{-\mathrm{i}aQ}\mathrm{e}^{-\mathrm{i}bP}\mathrm{e}^{\mathrm{i}\hbar ab}. \tag{3}$$

In the case of MTG's the rôles of Q and P are played by components, π^c_x and π^c_y respectively, of the vector operator (cf. [13])

$$\boldsymbol{\pi}^c = \mathbf{p} - \frac{e}{c}\mathbf{A} \tag{4}$$

(which has a meaning of the center of the Landau orbit [23]), where $\mathbf{A}$ is the vector potential of the magnetic field[2] $\mathbf{H}$ and $\mathbf{p}$ is the momentum operator. It is easy to check that for $\mathbf{H}=[0, 0, H]$

$$[\pi^c_y, \pi^c_x] = -\mathrm{i}\hbar\frac{eH}{c} \tag{5}$$

and substituting

$$Q = \frac{c}{eH}\pi^c_x, \qquad P = \pi^c_y \tag{6}$$

the relation (2) is revived. Magnetic translation operators (for the sake of simplicity the square lattice, determined by orthogonal vectors $\mathbf{a}_x=[a, 0]$ and $\mathbf{a}_y=[0, a]$, is considered) can be introduced as

$$T(n_\xi\mathbf{a}_\xi) = \exp(-\mathrm{i}n_\xi a\pi^c_\xi/\hbar), \qquad \xi = x, y. \tag{7}$$

Using the formulae (1) and (6) one can also write

$$T(n_x\mathbf{a}_x) = \mathrm{e}^{-\mathrm{i}n_x aeHQ/c\hbar} = \mathcal{Q}(n_x aeH/c\hbar); \tag{8a}$$
$$T(n_y\mathbf{a}_y) = \mathrm{e}^{-\mathrm{i}n_y aP/\hbar} = \mathcal{P}(n_y a/\hbar). \tag{8b}$$

Therefore, the commutation rule (3) yields

$$\begin{aligned} T(n_x\mathbf{a}_x)T(n_y\mathbf{a}_y) &= T(n_y\mathbf{a}_y)T(n_x\mathbf{a}_x)\exp(-\mathrm{i}n_x n_y a^2 eH/c\hbar) \\ &= T(n_y\mathbf{a}_y)T(n_x\mathbf{a}_x)\exp[-\mathrm{i}(n_x\mathbf{a}_x \times n_y\mathbf{a}_y)\cdot\mathbf{H}e/c\hbar]. \end{aligned} \tag{9}$$

Since $[\pi^c_y, \pi^c_x]$ is a complex number, the relation

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+[A,B]/2}$$

can be used and for $\mathbf{t}=[n_x a, n_y a]$ one obtains

$$T(\mathbf{t}) = \exp(-\mathrm{i}\mathbf{t}\cdot\boldsymbol{\pi}^c/\hbar) = T(n_x\mathbf{a}_x)T(n_y\mathbf{a}_y)\exp[\tfrac{1}{2}\mathrm{i}n_x n_y a^2 He/c\hbar]. \tag{10}$$

This formula defines a projective representation of a (two-dimensional) translation group, which in the absence of a magnetic field reduces to a pure translation operator acting on a function $\psi(\mathbf{r})$ in a standard way, i.e. (cf. also [6] and [16, Chaps. 1 and 8])

$$T(\mathbf{t})|_{\mathbf{A}=\mathbf{0}} = \exp(-\mathrm{i}\mathbf{t}\cdot\mathbf{p}/\hbar) \quad \text{and} \quad T(\mathbf{t})|_{\mathbf{A}=\mathbf{0}}\psi(\mathbf{r}) = \psi(\mathbf{r}-\mathbf{t}). \tag{11}$$

As it was mentioned in Sec. 1 consideration of projective representations can be replaced by investigation of a covering group and its vector representattions [15, 16]. Zak [7, 8, 13] defined a covering group $\mathcal{T}$ of the translation group T as a set of the following operators[3]

$$\tau(\mathbf{t} \mid \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_l) = \exp(-\mathrm{i}\mathbf{t}\cdot\boldsymbol{\pi}^c/\hbar)\exp[-\mathrm{i}e\Phi(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_l)/c\hbar], \tag{12}$$

[2] The gauge $\mathbf{A} = (\mathbf{H} \times \mathbf{r})/2$ is used in this work.

[3] Note that in the presented definition the sign is changed as compared with Zak's paper [7] in order to obtain later equations consitent with (11) and, moreover, with other works of Zak (e.g. [13]).

where $\mathbf{t}$ is a lattice vector, $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_l$ is any path joining the origin O with the point defined by $\mathbf{t}$ (i.e. $\mathbf{t} = \sum_{j=1}^{l} \mathbf{t}_j$; all $\mathbf{t}_j$, $1 \leq j \leq l$, are lattice vectors), and $\Phi(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_l)$ is the magnetic flux through the polygon enclosed by the vectors $\mathbf{t}_1, \ldots, \mathbf{t}_l, -\mathbf{t}$, i.e. $\Phi = \mathbf{H} \cdot \mathbf{S}$, with $\mathbf{S}$ being the area of the mentioned polygon. This area can be calculated as (see [7])

$$\mathbf{S} = \tfrac{1}{2}(\mathbf{t}_1 \times \mathbf{t}_2 + \ldots + \mathbf{t}_1 \times \mathbf{t}_l + \mathbf{t}_2 \times \mathbf{t}_3 + \ldots + \mathbf{t}_{l-1} \times \mathbf{t}_l), \tag{13}$$

i.e. by calculating all $\binom{n}{2}$ vector products (in the order determined by the order of vectors in the path).

On the other hand the Hamiltonin for an electron in a periodic potential $V(\mathbf{r})$ and a uniform magnetic field (described by the vector potential $\mathbf{A}$) is given as [6, 7]

$$\mathcal{H} = \frac{1}{2m}\boldsymbol{\pi}^2 + V(\mathbf{r}), \tag{14}$$

where

$$\boldsymbol{\pi} = \mathbf{p} + \frac{e}{c}\mathbf{A} \tag{15}$$

is the (vector) operator of the kinetic momentum. The operators defined in (12) commute with this Hamiltonian if the vector potential $\mathbf{A}$ fulfils the condition

$$\partial A_\xi / \partial \chi + \partial A_\chi / \partial \xi = 0; \qquad \text{for } \xi, \chi = x, y, z. \tag{16}$$

This relation holds, for example, for the gauge $\mathbf{A} = (\mathbf{H} \times \mathbf{r})/2$, which was used by both authors [6, 7] and will be applied in this work.

In his first paper Zak showed that

$$\begin{aligned} \tau(\mathbf{t} \mid \mathbf{t}_1, \ldots, \mathbf{t}_l)\tau(\mathbf{t}' \mid \mathbf{t}'_1, \ldots, \mathbf{t}'_j) &= \tau(\mathbf{t}+\mathbf{t}' \mid \mathbf{t}_1, \ldots, \mathbf{t}_l, \mathbf{t}'_1, \ldots, \mathbf{t}'_j); \qquad (17) \\ \tau(\mathbf{t} \mid \mathbf{t}_1, \ldots, \mathbf{t}_l)\tau(\mathbf{t}' \mid \mathbf{t}'_1, \ldots, \mathbf{t}'_j) &= \tau(\mathbf{t}' \mid \mathbf{t}'_1, \ldots, \mathbf{t}'_j)\tau(\mathbf{t} \mid \mathbf{t}_1, \ldots, \mathbf{t}_l). \\ &\quad \times \exp\left[-\mathrm{i}e(\mathbf{t} \times \mathbf{t}') \cdot \mathbf{H}/c\hbar\right]. \qquad (18) \end{aligned}$$

These relations yield that elements $(\mathbf{0} \mid \mathbf{t}_1, \ldots, \mathbf{t}_l)$ (with $\sum_{j=1}^{l} \mathbf{t}_j = \mathbf{0}$) belong to the center $Z(\mathcal{T})$. Moreover, the quotient group $\mathcal{T}/Z(\mathcal{T})$ is isomorphic with the group T of (ordinary) translations [7], and one can choose as representatives of *right* cosets elements $\tau(\mathbf{t} \mid \mathbf{t})$. It is easy to show that for any lattice vector $\mathbf{t}$ the operators $\tau(\mathbf{t} \mid \mathbf{t})$ and $\tau(\mathbf{t} \mid \mathbf{t}_1, \mathbf{t}_2)$, where $\mathbf{t}_1 + \mathbf{t}_2 = \mathbf{t}$ belong to the same right coset

$$\tau(\mathbf{0} \mid \mathbf{t}_1, \mathbf{t}_2, -\mathbf{t})\tau(\mathbf{t} \mid \mathbf{t}) = \tau(\mathbf{t} \mid \mathbf{t}_1, \mathbf{t}_2, -\mathbf{t}, \mathbf{t}) = \tau(\mathbf{t} \mid \mathbf{t}_1, \mathbf{t}_2). \tag{19}$$

Therefore, the MTG is a central extension of T by $Z(\mathcal{T})$ with a factor system $m(\mathbf{t}, \mathbf{t}')$ resulting from multiplication of coset representattives (cf. (19), [15, 24])

$$\tau(\mathbf{t} \mid \mathbf{t})\tau(\mathbf{t}' \mid \mathbf{t}') = \tau(\mathbf{t}+\mathbf{t}' \mid \mathbf{t}, \mathbf{t}') = \tau(\mathbf{0} \mid \mathbf{t}, \mathbf{t}', -(\mathbf{t}+\mathbf{t}'))\tau(\mathbf{t}+\mathbf{t}' \mid \mathbf{t}+\mathbf{t}'), \tag{20}$$

so

$$m(\mathbf{t}, \mathbf{t}') = \tau(\mathbf{0} \mid \mathbf{t}, \mathbf{t}', -(\mathbf{t}+\mathbf{t}')) = \exp\left[-\tfrac{1}{2}\mathrm{i}e(\mathbf{t} \times \mathbf{t}') \cdot \mathbf{H}/c\hbar\right]. \tag{21}$$

Therefore, (vector) representations of $\mathcal{T}$ can be constructed as products $\Gamma(z, \mathbf{t}) = \Delta(z)\Lambda(\mathbf{t})$, where Δ is a representation of the center Z and Λ is a projective representation of T with a factor system [16, 24]

$$\nu(\mathbf{t}, \mathbf{t}') = \Delta(m(\mathbf{t}, \mathbf{t}')). \tag{22}$$

Such a representation for $\Delta(z) = z$ is provided by a mapping introduced by Brown [6]

$$\Lambda(\mathbf{t}) = \exp(-\mathrm{i}\boldsymbol{\pi}^c \cdot \mathbf{t}/\hbar). \tag{23}$$

Therefore, this projective representation realize only one possible choice of Δ. Considerations of all (irreducible) representations allow us to get deep insight into algebraic structure of MTG's and physical relevance of their representations.

All these above mentioned facts suggest that MTG's can be considered as central extensions of T by G, where $T \simeq \mathbb{Z}^3$ is the translation group and $G \simeq Z(\mathcal{T})$ is a group of factors, so $G \subset \mathsf{U}(1)$. However, in the next section the central extension of $\mathbb{Z}^n$ by $\mathsf{U}(1)$ will be investigated. The first change ($\mathbb{Z}^3 \to \mathbb{Z}^n$) will allow us to investigate a general case of n-dimensional crystal lattice, where the second change ($G \to \mathsf{U}(1)$) is done for the sake of simplicity and clarity of considerations. It will then occur which factors $\mathrm{e}^{\mathrm{i}\phi} \in \mathsf{U}(1)$ form the center of $\mathcal{T}$.

3 Central Extensions of $\mathbb{Z}^n$ by $\mathsf{U}(1)$

It is easy to notice that any sequence $\mathbf{t}_1, \ldots, \mathbf{t}_j, -(\mathbf{t}_1 + \ldots + \mathbf{t}_j)$ corresponds to a loop 'drawn' in a crystal lattice (i.e. using lattice vectors), so it is a special case of a path $(\mathbf{t}_1, \ldots, \mathbf{t}_l)$ (cf. (12)). From the group-theoretical point of view the set of all paths is a free group F generated by all non-zero lattice vectors $\mathbf{t} \in T$. On the other hand, all loops form the kernel R of a homomorphism $M\colon F \to T$, which simply 'calculates' the value of a path in T, i.e.

$$M(\mathbf{t}_1, \ldots, \mathbf{t}_l) = \mathbf{t}_1 + \ldots + \mathbf{t}_l. \tag{24}$$

Moreover, each path can be written as a product (in the group F) of a loop and a one-element path $(\mathbf{t})$, which is chosen as the representative of the right-coset in the decomposition $F = \bigcup_{\mathbf{t}} R(\mathbf{t})$. These facts indicate close relations of Zak's approach with the Mac Lane method for determination all nonequivalent extensions of given groups. This method can be used to determine the second cohomology group $H^2(T, G)$ in a general case, i.e. for a given action of T on G. In the particular case, considered in this paper, one is interested in (nonequivalent) central extensions so this action is trivial. A detailed description of this method can be found in the original works of Mac Lane [17, 18] or in some other books and review articles (see [24, 25] or [19] and references quoted therein). Many examples of its application can be found elsewhere [26, 27]. Among others it has been applied to thorough investigation of MTG's in two dimensions [28].

The main idea of the Mac Lane method consists in replacing an exact sequence

$$\{1\} \longrightarrow G \longrightarrow \mathcal{T} \longrightarrow T \longrightarrow \{0\} \tag{25}$$

by the following one

$$\{1\} \longrightarrow R \longrightarrow F \longrightarrow T \longrightarrow \{0\}. \tag{26}$$

These sequences are related with each other by a family[4] of the so-called operator homomorphisms $\phi\colon R \to G$.

In the following subsections the Mac Lane method is realized step by step in the case $T \simeq \mathbb{Z}^n$ and $G = \mathsf{U}(1)$. Lattice vectors $\mathbf{t}$ will be hereafter replaced by n-tuples $\mathbf{k} := (k_1, \ldots, k_n)$, $k_j \in \mathbb{Z}$ due to the obvious isomorphism

$$(k_1, k_2, \ldots, k_n) = \mathbf{k} \leftrightarrow \mathbf{t}_{\mathbf{k}} = \sum_{j=1}^{n} k_j \mathbf{a}_j, \tag{27}$$

where $\{\mathbf{a}_j\}_{j=1,\ldots,n}$ is a crystal basis for a given lattice.

3.1 Generators of $\mathbb{Z}^n$

The very first step in the procedure is a choice of generators of the group $\mathbb{Z}^n$, i.e. the translation group T. In this paper the most natural set of generators is used, i.e.

$$A_T := \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \tag{28}$$

consists of n linearly independent vectors, which form a crystal basis. As the generators of $\mathbb{Z}^n$ we choose n-tuples

$$\mathbf{e}_j := (0, \ldots, 0, k_j = 1, 0, \ldots, 0), \qquad A := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}, \tag{29}$$

related with vectors $\mathbf{a}_j$ by the isomprphism (27). Now a free group F of rank n has to be introduced. The alphabet X of this group consists of n letters x_j, $1 \leq j \leq n$, such that

$$M(x_j) = \mathbf{e}_j. \tag{30}$$

This formula determines an epimorphism $M\colon F \to \mathbb{Z}^n$, since each m-letter word $f^{(m)} \in F$ can be written as

$$f^{(m)} = \prod_{k=1}^{m} \xi_k^{\varepsilon_k}, \qquad \xi_k \in X,\ \varepsilon_k = \pm 1, \tag{31}$$

[4] Strictly speaking these mappings form a group, denoted by $\mathrm{Hom}_F(R, G)$, with the point-wise composition rule $(\phi + \phi')(r) = \phi(r)\phi'(r)$.

and

$$M(f^{(m)}) = \sum_{k=1}^{m} \varepsilon_k M(\xi_k).$$

In the further considerations the inverse of any word $f \in F$ will be denoted as $\bar{f}$ and, of course, it will be also applied to the letters $x_j \in X$, so $x_j^{-1} = \bar{x}_j$.

3.2 Decomposition of F

The kernel Ker $M := R$ consits of such words $f \in F$ that $M(f) = \mathbf{0} := (0, \ldots, 0)$ and the group F can be decomposed into *right* cosets with respect to R; since these cosets are counter-images of M then they can be labelled by $\mathbf{k} \in \mathbb{Z}^n$, so

$$F = \bigcup_{\mathbf{k}} R f_{\mathbf{k}}, \tag{32}$$

where $M(f_{\mathbf{k}}) = \mathbf{k}$. One may choose any representatives $f_{\mathbf{k}}$, but in the presented procedure it is important that a set of representatives

$$S := \{f_{\mathbf{k}} \mid \mathbf{k} \in \mathbb{Z}^n\} \tag{33}$$

is the Schreier set. It means that (cf. [19, 24, 25])

$$1_F \in S, \qquad X \subset S; \tag{34a}$$

$$\left(f^{(m)} \neq 1_f,\ f^{(m)} \in S\right) \Longrightarrow \left(f^{(m')} \in S, \quad \forall\, m' = 0, 1, \ldots, m-1,\right) \tag{34b}$$

where $f^{(m')}$ denotes the m'-letter initial subword of $f^{(m)}$, i.e. (cf. (31))

$$f^{(m')} = \prod_{k=1}^{m'} \xi_k^{\varepsilon_k}; \quad f^{(0)} := 1_F. \tag{35}$$

The conditions (34) are satisfied by the following set

$$S := \{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \mid \mathbf{k} \in \mathbb{Z}^n\}. \tag{36}$$

This definition determines also a mapping $\Psi\colon \mathbb{Z}^n \to F$ such that

$$\Psi(\mathbf{k}) = \Psi\Big(\sum_{j=1}^{n} k_j \mathbf{e}_j\Big) = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}; \quad (M \circ \psi)(\mathbf{k}) = \mathbf{k}. = f_{\mathbf{k}}. \tag{37}$$

A composition of mappings Ψ and M in the opposite order is the so-called choice function $\beta := \Psi \circ M$, which maps each $f \in F$ onto the corresponding coset representative $f_{\mathbf{k}}$, where $\mathbf{k} = M(f)$ and $f \in R\, f_{\mathbf{k}}$. It is evident that each word $f \in F$ can be written as a product of q words from the Schreier set (36), i.e.

$$f^{(m)} = \prod_{l=1}^{q} x_1^{k_1^{(l)}} \ldots x_n^{k_n^{(l)}}, \qquad \sum_{l=1}^{q} \sum_{j=1}^{n} |k_j^{(l)}| = m. \tag{38}$$

Using this form of $f^{(m)}$ the choice function β is defined as

$$\beta\Big(f^{(m)}\Big) = x_1^{\kappa_1} x_2^{\kappa_2} \ldots x_n^{\kappa_n}, \qquad \kappa_j = \sum_{l=1}^{q} k_j^{(l)}. \tag{39}$$

3.3 Alphabet of R

The next very important step is to find all factors $\varrho\colon \mathbb{Z}^n \times \mathbb{Z}^n \to R$ determined as

$$\varrho(\mathbf{k}, \mathbf{k}') := f_{\mathbf{k}} f_{\mathbf{k}'} \bar{f}_{\mathbf{k}+\mathbf{k}'}, \tag{40}$$

which can be also written as (recall that M restricted to S is a bijection)

$$\varrho(s, s') = s s' \overline{\beta(s s')}, \quad s, s' \in S. \tag{41}$$

It is easy to notice that this factor system is normalized since

$$\varrho(1_F, s) = \varrho(s, 1_F) = 1_F \qquad \text{or} \qquad \varrho(\mathbf{0}, \mathbf{k}) = \varrho(\mathbf{k}, \mathbf{0}) = 1_F.$$

All factors $\varrho(s, s')$ will be necessary in the last step of this procedure but now one needs only a part of them. The Nielsen–Schreier theorem (see, e.g., [25]) states that the alphabet Y of the free group $R \subset F$ consists of nontrivial factors obtained for $s' \in X$ (i.e. for $\mathbf{k}' \in A$, see Eq. (29)). It means that

$$\begin{aligned} Y &:= \{y = s\xi\overline{\beta(s\xi)} \mid s \in S,\ \xi \in X,\ y \neq 1_F\} && \text{(42a)} \\ &= \{y = f_{\mathbf{k}} f_{\mathbf{k}'} \bar{f}_{\mathbf{k}+\mathbf{k}'} \mid \mathbf{k} \in \mathbb{Z}^n,\ \mathbf{k}' \in A,\ y \neq 1_F\}. && \text{(42b)} \end{aligned}$$

It is easy to notice that there are $(n-1)$ *types* of such factors that describe transformations, which are necessary to find a representative for any word $f \in F$. It follows from the fact that the letters of the alphabet Y are determined in the procedure when the letters x_j, $j=1,\ldots,n-1$ are moved form the end of a word $sx_j = x_1^{k_1} \ldots x_d^{k_d} x_j$ to the 'proper', i.e. the j-th, position. Therefore, these letters are connected with $(n-1)$ cyclic permutations

$$(12\ldots n),\ (23\ldots n), \ldots, (n-1\, n)$$

(for a given $j=1,2,\ldots,n-1$ the last $n+1-j$ symbols $x_l^{k_l}$ in the word s have to permuted in the cyclic order to 'catch' the added x_j). The letters of the alphabet Y will be hereafter denoted as $A_j^{(\mathbf{k})}$, where $\mathbf{k} := (k_1, \ldots, k_n)$, and can be calculated as

$$\begin{aligned} A_j^{(\mathbf{k})} &= x_1^{k_1} \ldots x_j^{k_j} \ldots x_n^{k_n} x_j \left(x_1^{k_1} \ldots x_j^{k_j+1} \ldots x_n^{k_n} \right)^{-1} \\ &= x_1^{k_1} \ldots x_j^{k_j} \ldots x_n^{k_n} x_j \bar{x}_n^{k_n} \ldots \bar{x}_j^{k_j+1} \ldots \bar{x}_1^{k_1}. && \text{(43)} \end{aligned}$$

It is important to notice that for a given j not all n-tuples $\mathbf{k}$ determine nontrivial factors ϱ (i.e. letters $A_j^{(\mathbf{k})}$). If all exponents k_l for $l>j$ are equal to zero than the corresponding factor $A_j^{(\mathbf{k})} = 0$. It means that for a given j all n-tuples $\mathbf{k} = (k_1, \ldots, k_j, 0, \ldots, 0)$ have to be omitted. It will occur later on that a set of parameters labelleling (nonequivalent) extensions depends only on numbers $k_{j+1}, \ldots, k_n$ (for each $j=1,2,\ldots,n-1$).

Two remarks are in place. At first, it is evident that the alphabet Y contains more (much more, in the considered case) letters than the alphabet X. Secondly, one can compare obtained results with those for a finite translation group $T \simeq \bigotimes_{j=1}^n \mathbb{Z}_{N_j}$. In this case n-tuples $(k_1, \ldots, k_l, 0, \ldots, 0)$ are omitted only for $l<j$. For $l=j$ one has to consider all $\mathbf{k}$'s with $k_j = N-1$, which lead to letters $A_j^{(\mathbf{k})}$ of the following form

$$A_j^{(k_1,\ldots,k_{j-1},N_j-1,0,\ldots,0)} = x_1^{k_1} \ldots x_{j-1}^{k_{j-1}} x_j^{N_j} \bar{x}_{j-1}^{k_{j-1}} \ldots \bar{x}_1^{k_1}.$$

These letters appear in the considerations due to the relations, for finite cyclic groups $\mathbb{Z}_{N_j}$ or letters x_j,

$$N_j \mathbf{a}_j = \mathbf{0} \quad \text{or} \quad \beta(x_j^{N_j}) = 1_F.$$

However, in the above formulae a more general form of Eq. (43) was used

$$A_j^{(\mathbf{k})} = x_1^{k_1} \ldots x_j^{k_j} \ldots x_n^{k_n} x_j \left(\beta(x_1^{k_1} \ldots x_j^{k_j+1} \ldots x_n^{k_n}) \right)^{-1}. \tag{44}$$

It is easy to determine a number of letters $A_j^{(\mathbf{k})}$ (for the finite translation group T) as

$$\sum_{j=1}^{n} \left[\left(\prod_{l=1}^{n} N_l \right) - \left(\prod_{l=1}^{j-1} N_l \right) (N_j - 1) \right] = (n-1) \left(\prod_{l=1}^{n} N_l \right) + 1$$

what agrees with the Nielsen-Schreier theorem, which reads

$$|Y| = (|X| - 1)|T| + 1. \tag{45}$$

Any element of the kernel R can be written using letters of the alphabet Y; however, in actual calculations the alphabet X is rather used, so one has to 'translate' obtained words into the proper

alphabet. The appropriate formula has also been given by the Nielsen–Schreier theorem. A given word $f^{(m)} \in R$ (see Eq. (31)) can be written as

$$f^{(m)} = \prod_{k=1}^{m} \beta(f^{(k-1)})\xi_k^{\varepsilon_k}\overline{\beta(f^{(k)})}, \tag{46}$$

where subwords $f^{(k)}$ are determined in (35) (cf. also definition (42a)).

It is convenient to introduce another set of letters (anothe alphabet in R), which are simply products of letters $A_j^{(\mathbf{k})}$ defined in the following way

$$r_j^{(\mathbf{k})} := \begin{cases} \prod_{l=0}^{k_j-1} A_j^{(k_1,\ldots,k_{j-1},l,k_{j+1},\ldots,k_n)}, & \text{for } k_j > 0; \\ 1_F, & \text{for } k_j = 0; \\ \prod_{l=-1}^{k_j} \bar{A}_j^{(k_1,\ldots,k_{j-1},l,k_{j+1},\ldots,k_n)} & \\ = \left(\prod_{l=k_j}^{-1} A_j^{(k_1,\ldots,k_{j-1},l,k_{j+1},\ldots,k_n)}\right)^{-1}, & \text{for } k_j < 0. \end{cases} \tag{47}$$

On the other hand

$$A_j^{(\mathbf{k})} = \bar{r}_j^{(\mathbf{k})}\, r_j^{(\mathbf{k}+\mathbf{e}_j)}. \tag{48}$$

Considerations concernig the number of letters r (also for finite translations groups) are identical as for the letters A. It is easy to check (see the appendix) that

$$r_j^{(\mathbf{k})} = x_1^{k_1}\ldots x_{j-1}^{k_{j-1}}\, x_{j+1}^{k_{j+1}}\ldots x_n^{k_n}\, x_j^{k_j}\left(\beta(x_1^{k_1}\ldots x_n^{k_n})\right)^{-1}. \tag{49}$$

These letters can be generalized to all permutations $\sigma \in S_n$ of the indices $1, 2, \ldots, n$, which are defined by the following formula (cf. (49) and (A.1))

$$r_\sigma^{(\mathbf{k})} = x_{\sigma(1)}^{k_{\sigma(1)}}\ldots x_{\sigma(n)}^{k_{\sigma(n)}}\left(x_1^{k_1}\ldots x_n^{k_n}\right)^{-1}. \tag{50}$$

Of course we have

$$r_j^{(\mathbf{k})} = r_{(j\, j+1\, \ldots, n)}^{(\mathbf{k})}.$$

3.4 Action of F on R

Now we have to consider conjugation of letters $A_j^{(\mathbf{k})} \in Y$ by letters $x_q \in X$, i.e. automorphisms of R, which are inner ones in F. Obtained words, obviously belonging to R, can be expressed in the alphabet Y, but it is more convenient to use both sets: letters $A_j^{(\mathbf{k})}$ and $r_\sigma^{(\mathbf{k})}$ (note that the latter set is not an alphabet; only letters $r_j^{(\mathbf{k})}$ corresponding to the cyclic permutations form an alphabet). It can be shown that

$$x_q A_j^{(\mathbf{k})} \bar{x}_q = \begin{cases} r_{(q\, q-1\, q-2\, \ldots, 1)}^{(k_1,\ldots,k_{q-1},1,0,\ldots,0)} A_j^{(\mathbf{k}+\mathbf{e}_q)} \bar{r}_{(q\, q-1\, q-2\, \ldots, 1)}^{(k_1,\ldots,k_{q-1},1,0,\ldots,0)}, & \text{for } j \geq q; \\ r_{(q\, q-1\, q-2\, \ldots, 1)}^{(k_1,\ldots,k_{q-1},1,0,\ldots,0)} A_j^{(\mathbf{k}+\mathbf{e}_q)} \bar{r}_{(q\, q-1\, q-2\, \ldots, 1)}^{(k_1,\ldots,k_j+1,\ldots,k_{q-1},1,0,\ldots,0)}, & \text{for } j < q. \end{cases} \tag{51}$$

The symbols $r_\sigma^{(\mathbf{k})}$ should be rewritten using letters $r_j^{(\mathbf{k})}$ or $A_j^{(\mathbf{k})}$, however we will see later on that this form is much better for further calculations.

3.5 Operator homomorphisms

A homomorphism $\phi\colon R \to \mathsf{U}(1)$ is called operator homomorphism if the following diagram is commutative:

$$\begin{array}{ccc} R & \xrightarrow{\Xi_f} & R \\ \phi\ \downarrow & & \downarrow\ \phi \\ \mathsf{U}(1) & \xrightarrow{\Delta(M(f))} & \mathsf{U}(1) \end{array}, \tag{52}$$

where $\Xi_f(r) = fr\bar{f}$ and $\Delta\colon T \to \mathrm{Aut}\,\mathsf{U}(1)$ is a homomorphism describing an action of T on $\mathsf{U}(1)$. We consider the trivial action Δ, so the operator homomorphisms satisfy condition

$$\phi(r) = \phi(fr\bar{f}), \quad r \in R,\ f \in F. \tag{53}$$

It suffices to check this condition only for the alphabets X and Y, since ϕ is a homomorphism, so the relation (51) will be used. To simplify notation the image of a letter $A_j^{(\mathbf{k})}$ will be hereafter denoted by $a_j^{(\mathbf{k})} = \phi(A_j^{(\mathbf{k})})$. Since the group of factors $\mathsf{U}(1)$ is abelian then for $j \geq q$ one obtains

$$a_j^{(\mathbf{k})} = a_j^{(\mathbf{k}+\mathbf{e}_q)}. \tag{54}$$

It means that $a_j^{(\mathbf{k})}$ depends only on entries k_q for $j < q$. In the special case we have that $a_{n-1}^{(\mathbf{k})}$ depends only on k_n. The definition (47) gives

$$\phi(r_j^{(\mathbf{k})}) = (a_j^{(\mathbf{k})})^{k_j}. \tag{55}$$

It is easy to prove that

$$r_{(q\, q-1\, q-2\, \ldots, 1)}^{(k_1,\ldots,k_{q-1},1,0,\ldots,0)} = \prod_{l=1}^{q-1} r_l^{(k_1,\ldots,k_l,0,\ldots 0,k_q=1,0,\ldots,0)} \tag{56}$$

so

$$\phi\Big(r_{(q\, q-1\, q-2\, \ldots, 1)}^{(k_1,\ldots,k_{q-1},1,0,\ldots,0)}\Big) = \prod_{l=1}^{q-1} \Big(a_l^{(k_1,\ldots,k_l,0,\ldots 0,k_q=1,0,\ldots,0)}\Big)^{k_l}. \tag{57}$$

Introducing a parameter (recall that we are interested in $j < q$ now)

$$a_{j,q} := a_j^{(0,\ldots,0,k_q=1,0,\ldots,0)} \tag{58}$$

and taking into account relation (54) the above formula can be written as

$$\phi\Big(r_{(q\, q-1\, q-2\, \ldots, 1)}^{(k_1,\ldots,k_{q-1},1,0,\ldots,0)}\Big) = \prod_{l=1}^{q-1} a_{l,q}^{k_l}. \tag{59}$$

Therefore, the second equation in (51) leads to the following condition

$$a_j^{(\mathbf{k})} = a_j^{(k_1,\ldots,k_q+1,\ldots,k_n)} \Big(\prod_{l=1}^{q-1} a_{l,q}^{k_l}\Big) \Big(\prod_{l=1}^{q-1} a_{l,q}^{-k_l}\Big) a_{j,q}^{-1} \tag{60}$$

for $j < q$. Finally, from both conditions (54) and (60) we obtain

$$a_j^{(\mathbf{k})} = \prod_{q=j+1}^{n} a_{j,q}^{k_q}. \tag{61}$$

Therefore, each operator homomorphism ϕ is determined by $\binom{n}{2}$ parameters $a_{j,q} \in \mathsf{U}(1)$. These numbers can be replaced by corresponding real parameters $\alpha_{j,q} \in [0, 2\pi)$ with $a_{j,q} = \exp(\mathrm{i}\alpha_{j,q})$. The last ones can be arranged in a real upper-triangular n-dimensional matrix A'

$$\mathsf{A}'_{j,q} := \begin{cases} \alpha_{j,q}, & \text{for } j < q; \\ 0, & \text{otherwise.} \end{cases} \tag{62}$$

It can be shown that for finite lattices there are n additional parameters corresponding to the periodic boundary conditions.

3.6 Crossed homomorphisms

The operator homomorphisms form a group denoted hereafter as $\mathrm{Hom}_F(R,T)$. The main result of Mac Lane is the following theorem:

$$H^2(T,\mathsf{U}(1)) = \mathrm{Hom}_F(R,\mathsf{U}(1))/Z^1_\Delta(F,\mathsf{U}(1))|_R, \tag{63}$$

where $Z^1_\Delta(F,\mathsf{U}(1))|_R$ is a group of one-cocycles (the so-called crossed homomorphisms) $Z^1_\Delta(F,\mathsf{U}(1))$, restricted to R, i.e.

$$Z^1_\Delta(F,\mathsf{U}(1)) := \{\gamma\colon F\to\mathsf{U}(1) \mid \gamma(ff') = \gamma(f)\,\Delta(M(f))[\gamma(f')]\}.$$

In the case of central extensions (i.e. trivial action Δ) it means that each mapping γ should be a homomorphism and, therefore, it is determined by values $\gamma(x_j)$, $x_j\in X$.

Since all operator homomorphisms are determined by parameters $a_{j,q}$ given by Eq. (58), then it is enough to calculate images of homomorphism ($j<q$)

$$\gamma\Big(A_j^{(0,\dots,0,k_q=1,0,\dots,0)}\Big) = \gamma(x_q x_j \bar{x}_q \bar{x}_j) = 1\in\mathsf{U}(1).$$

Therefore, all operator homomorphisms determine nonequivalent central extensions of $\mathbb{Z}^n$ by $\mathsf{U}(1)$.

3.7 Factor systems

The last task is to find factor systems, which are determined as

$$m'_\phi(\mathbf{k},\mathbf{k}') = \phi(\varrho(\mathbf{k},\mathbf{k}')), \tag{64}$$

where ϱ is defined in (40). It can be easily done if one express all factors ϱ by symbols $r_j^{(\mathbf{k})}$.

$$\varrho(\mathbf{k},\mathbf{k}') = (x_1^{k_1}\dots x_n^{k_n})(x_1^{k'_1}\dots x_n^{k'_n})(\bar{x}_n^{k_n+k'_n}\dots\bar{x}_1^{k_1+k'_1}).$$

Considering the product of the first two representatives $s\in S$ in the above formula it is convenient to introduce n-tuples

$$\mathbf{k}_l := (k_1+k'_1,\dots,k_l+k'_l,k_{l+1},\dots,k_n); \qquad \mathbf{k}_0 := \mathbf{k}, \quad \mathbf{k}_n = \mathbf{k}+\mathbf{k}'.$$

Using this symbols we can write

$$\begin{aligned}
\Big(\prod_{j=1}^n x_j^{k_j}\Big)\Big(\prod_{j=1}^n x_j^{k'_j}\Big) &= \bar{r}_1^{(\mathbf{k}_0)} r_1^{(\mathbf{k}_1)} x^{k_1+k'_1}\Big(\prod_{j=2}^n x_j^{k_{1,j}}\Big)\Big(\prod_{j=2}^n x_j^{k'_j}\Big)\\
&= \Big(\prod_{j=1}^q \bar{r}_j^{(\mathbf{k}_{j-1})} r_j^{(\mathbf{k}_j)}\Big)\Big(\prod_{j=1}^q x_j^{k_j+k'_j}\Big)\Big(\prod_{j=q+1}^n x_j^{k_{1,j}}\Big)\Big(\prod_{j=q+1}^n x_j^{k'_j}\Big)\\
&= \Big(\prod_{j=1}^{n-1} \bar{r}_j^{(\mathbf{k}_{j-1})} r_j^{(\mathbf{k}_j)}\Big)\Big(\prod_{j=1}^n x_j^{k_j+k'_j}\Big)
\end{aligned} \tag{65}$$

Therefore

$$\varrho(\mathbf{k},\mathbf{k}') = \prod_{j=1}^{n-1} \bar{r}_j^{(\mathbf{k}_{j-1})} r_j^{(\mathbf{k}_j)} \tag{66}$$

and

$$m'(\mathbf{k},\mathbf{k}') = \prod_{j=1}^{n-1}\phi(\bar{r}_j^{(\mathbf{k}_{j-1})})\phi(r_j^{(\mathbf{k}_j)}) = \prod_{j=1}^{n-1}\prod_{q=j+1}^{n} a_{j,q}^{k_q k'_j} = \exp\Big\{\mathrm{i}\sum_{j=1}^{n-1}\sum_{q=j+1}^{n}\alpha_{j,q}k_q k'_j\Big\}. \tag{67}$$

For a given factor system (a given matrix A') the central extension of $\mathbb{Z}^n$ by $\mathsf{U}(1)$ is a set of pairs $[\exp(\mathrm{i}\alpha),\mathbf{k}]$, where $\exp(\mathrm{i}\alpha)\in\mathsf{U}(1)$ and $\mathbf{k}=(k_1,k_2,\dots,k_n)$ with the following multiplication rule

$$[\exp(\mathrm{i}\alpha),\mathbf{k}][\exp(\mathrm{i}\alpha'),\mathbf{k}'] = \Big[\exp\{\mathrm{i}\alpha+\mathrm{i}\alpha'+\mathrm{i}\sum_{j=1}^{n-1}\sum_{q=j+1}^{n}\alpha_{j,q}k_q k'_j\},\mathbf{k}+\mathbf{k}'\Big]. \tag{68}$$

3.8 Some remarks

Before discussing obtained results in the special case $n=3$ (the most interesting in physical applications) it is possible to state some general properties of the derived formulae. At first, it has to be stressed that, in general, 'nonequivalent' does not mean 'nonisomorphic' and, therefore, some of obtained groups are isomorphic. For example, if one defines an automorphism of $\mathsf{U}(1)$ as

$$\exp(\mathrm{i}\alpha) \mapsto \exp(\mathrm{i}\eta\alpha), \quad \eta \in \mathbb{R},\ \eta \neq 0$$

then factors determined by matrices A' and $\eta\mathsf{A}'$ lead to isomporphic groups due to a mapping

$$[\exp(\mathrm{i}\alpha), \mathbf{k}] \mapsto [\exp(\mathrm{i}\eta\alpha), \mathbf{k}].$$

Applying the formula (68) we can calculate product of 'pure' translations, i.e. elements $[1, \mathbf{k}]$. It is very interesting to find this in the case of generators of $\mathbb{Z}^n$, i.e. $\mathbf{e}_j$. To be more precise, we calculate a product corresponding, under the isomorphism (27), to a loop constructed from vectors $\mathbf{a}_j$ and $\mathbf{a}_q$

$$[1, -\mathbf{e}_q][1, -\mathbf{e}_j][1, \mathbf{e}_q][1, \mathbf{e}_j] = [\exp(\mathrm{i}\alpha_{j,q}), -\mathbf{e}_q - \mathbf{e}_j][\exp(\mathrm{i}\alpha_{j,q}), \mathbf{e}_j + \mathbf{e}_q] = [\exp(\mathrm{i}\alpha_{j,q}), \mathbf{0}].$$

Therefore, the parameters of a given extension correspond with factors, which are gained after completing 'primitive' loops. Going in the oposite direction we obtain

$$[1, \mathbf{e}_j][1, \mathbf{e}_q][1, -\mathbf{e}_j][1, -\mathbf{e}_q] = [\exp(-\mathrm{i}\alpha_{j,q}), \mathbf{0}],$$

i.e. the inverse of the previous factor.

One more very important fact has to be pointed: the Mac Lane method does not provide us with the group of two-coboundaries $B^2(\mathbb{Z}^n, \mathsf{U}(1))$ and, therefore, we do not find trivial factor systems in an *explicite* way. It means that choosing generators (the set A) and the representatives (the set S) one obtains factor systems in a form, which, sometimes, may be inconvenient in the further applications. Hence, it may be necessary to find also (usually, by direct calculations) at least one element of the group $B^2(\mathbb{Z}^n, \mathsf{U}(1))$.

Trivial factor systems (in the case of central extensions) are determined by a normalized (i.e. $\psi(\mathbf{0})=1$) mappings $\psi\colon \mathbb{Z}^n \to \mathsf{U}(1)$ according to the following formula

$$\theta_\psi(\mathbf{k}, \mathbf{k}') := \psi(\mathbf{k})\psi(\mathbf{k}')/\psi(\mathbf{k}+\mathbf{k}'). \tag{69}$$

One can consider the following mapping[5] $\psi\colon \mathbb{Z}^n \to \mathsf{U}(1)$:

$$\psi(\mathbf{k}) = \exp\Big\{\frac{\mathrm{i}}{2}\sum_{j=1}^{n-1}\sum_{q=j+1}^{n} \alpha_{j,q} k_q k_j\Big\}.$$

This mapping defines a trivial factor system

$$\theta(\mathbf{k}, \mathbf{k}') = \exp\Big\{-\frac{\mathrm{i}}{2}\sum_{j=1}^{n-1}\sum_{q=j+1}^{n} \alpha_{j,q}(k_q k'_j + k'_q k_j)\Big\}. \tag{70}$$

A product of the factor system (67) and the trivial one (70) determines an equivalent extension with the following factor system

$$m(\mathbf{k}, \mathbf{k}') = m'(\mathbf{k}, \mathbf{k}')\theta(\mathbf{k}, \mathbf{k}') = \exp\Big\{-\frac{\mathrm{i}}{2}\sum_{j=1}^{n-1}\sum_{q=j+1}^{n} \alpha_{j,q}(k_j k'_q - k_q k'_j)\Big\}.$$

Introducing matrices (tensors):

$$\mathsf{A} \;=\; \mathsf{A}' - (\mathsf{A}')^{\mathrm{T}}; \quad \mathsf{A}_{j,q} = \begin{cases} \alpha_{j,q}, & \text{for } j < q; \\ 0, & \text{for } j = q; \\ -\alpha_{q,j}, & \text{for } j > q \end{cases} \tag{71a}$$

$$(\mathbf{k} \otimes \mathbf{k}')_{j,q} \;=\; k_j k'_q \tag{71b}$$

[5]For a finite gauge group G it may be impossible to find such $b \in G$ that $b^2 = a$ for each $a \in G$.

this factor system can be written as

$$m(\mathbf{k},\mathbf{k}') = \exp\Big\{-\frac{\mathrm{i}}{2}\mathsf{A}\cdot(\mathbf{k}\otimes\mathbf{k}')\Big\}, \tag{72}$$

where

$$A\cdot B := \prod_{j,q=1}^{n} A_{j,q}B_{j,q}$$

is a scalar product of matrcies (a transvection of tensors [29]).

It is obvious that obtained factor systems are periodic with respect to the parameters $\alpha_{j,q}$ with identical periods 4π. Comparing the obtained formula with Eq. (21) one can notice that $(\mathbf{k}\otimes\mathbf{k}')/2$ corresponds to $(\mathbf{t}\times\mathbf{t}')/2$, i.e. the area of a triangle determined by vectors $\mathbf{t}$ and $\mathbf{t}'$, whereas the antisymmetric tensor A corresponds (up to multiplicative constants) to the external magnetic field $\mathbf{H}$. It will be more clear considering the case $n=3$, when three-dimensional vectors will be associated with the introduced tensors.

3.9 Special case: $n = 3$

The final formula (72) of the previous section can be written down in an explicite way for $n=3$ in the following form

$$m(\mathbf{k},\mathbf{k}') = \exp\Big\{-\frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \alpha_{1,2} & \alpha_{1,3} \\ -\alpha_{1,2} & 0 & \alpha_{2,3} \\ -\alpha_{1,3} & -\alpha_{2,3} & 0 \end{pmatrix}\cdot\begin{pmatrix} k_1k_1' & k_1k_2' & k_1k_3' \\ k_2k_1' & k_2k_2' & k_2k_3' \\ k_3k_1' & k_3k_2' & k_3k_3' \end{pmatrix}\Big\}.$$

For both tensors one can find corresponding (three-dimensional) vectors $\mathbf{h}$ and $\mathbf{s}$, respectively, using the symbol ε_{jqp} (i.e. the three-dimensional Ricci symbol [29]), namely

$$\begin{aligned} h_j &= \tfrac{1}{2}\sum_{q,p}\varepsilon_{jqp}\alpha_{q,p}; \\ s_j &= \tfrac{1}{2}\sum_{q,p}\varepsilon_{jqp}k_qk_p', \end{aligned}$$

so

$$\begin{aligned} \mathbf{h} &= [\alpha_{2,3}, -\alpha_{1,3}, \alpha_{1,2}]; \\ \mathbf{s} &= \tfrac{1}{2}[k_2k_3' - k_3k_2', k_3k_1' - k_1k_3', k_1k_1' - k_2k_1'] \end{aligned}$$

and

$$m(\mathbf{k},\mathbf{k}') = \exp\{-\mathrm{i}\mathbf{h}\cdot\mathbf{s}\}, \tag{73}$$

where $\mathbf{h}\cdot\mathbf{s}$ denotes a scalar product of vectors. Comparing $\mathbf{s}$ with

$$\tfrac{1}{2}(\mathbf{t}_{\mathbf{k}}\times\mathbf{t}_{\mathbf{k}'}) = s_1(\mathbf{a}_2\times\mathbf{a}_3) + s_2(\mathbf{a}_3\times\mathbf{a}_1) + s_3(\mathbf{a}_1\times\mathbf{a}_2)$$

one can see that $\mathbf{s}$ can be considered as a covariant vector corresponding with the area $\mathbf{S}=\frac{1}{2}(\mathbf{t}_{\mathbf{k}}\times\mathbf{t}_{\mathbf{k}'})$. On the other hand $\mathbf{h}$ is a contravariant vector written in the basis $\{\mathbf{a}_j\}$ as

$$\mathbf{h} = \sum_{j=1}^{3} h_j\mathbf{a}_j = \frac{eV}{c\hbar}\mathbf{H}$$

i.e. $h_j = (eV/c\hbar)H_j$, where H_j is a component of the magnetic field $\mathbf{H}$ in the direction $\mathbf{a}_j$ and $V = \mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{a}_3)$. Substituing these formulae into Eq. (73) one obtains

$$m(\mathbf{k},\mathbf{k}') = \exp\Big\{-\mathrm{i}\frac{e}{c\hbar}\mathbf{H}\cdot\mathbf{S}\Big\}$$

what agrees with formula (21). Introducing $\omega = hc/e$ being the fluxon, i.e. the elementary quantum of magnetic flux (cf. [5, 7, 30, 31]) and replacing $\mathbf{k}$ and $\mathbf{k}'$ by the corresponding vectors $\mathbf{t}_{\mathbf{k}} \equiv \mathbf{t}$ and $\mathbf{t}_{\mathbf{k}'} \equiv \mathbf{t}'$ this formula may be writtens as

$$m(\mathbf{t},\mathbf{t}') = \exp\Big\{-2\pi\mathrm{i}\frac{\mathbf{H}\cdot\mathbf{S}}{\omega}\Big\}$$

what means that for a given vectors $\mathbf{t}$ and $\mathbf{t}'$ the corresponding factor $m(\mathbf{t},\mathbf{t}')$ is trivial if the total magnetic flux through the triangle $\mathbf{t},\mathbf{t}',-(\mathbf{t}+\mathbf{t}')$ is equal to integer number of fluxons and these factors are periodic in the magnetic field with periods

$$H_j = \frac{2}{V}\omega|\mathbf{a}_j|.$$

If components of the magnetic field $\mathbf{H}$ in the crystal basis $\{\mathbf{a}_j\}$ are rational numbers then number of different factors $m(\mathbf{t},\mathbf{t}')$ is finite.

4 Final remarks

The main result of this work is the relation (72) (and discussion below it), which show that a magnetic field can be interpreted as an $(n\times n)$ antisymmetric tensor and factor systems in n-dimensional MTG is determined by a transvection of tensors. However, this fact was not used in all its aspects, among others covariant and contravariant tensors were not introduced and discussed in details. Nevertheless, some important general conclusions can be formulated: (i) there is no possibility to introduce MTG in one dimension; (ii) a (contravariant) tensor of rank $n-2$ corresponds to A (see, e.g., [29]); (iii) in the special cases, $n=2,3$, it means that A can be represented by a scalar and a vector, respectively. It seems that investigations of MTG's in four dimensions allow us to introduce the electromagnetic tensor F, providing that we conisder discrete time, and to consider commutativity of time and space displacements.

It is also important to compare the (equivalent) factor systems (67) and (72). Considering a pair of primitive vectors, namely $\mathbf{a}_j,\mathbf{a}_q$ with $j<q$, we obtain

$$m'(\mathbf{a}_j,\mathbf{a}_q)=1, \qquad m'(\mathbf{a}_q,\mathbf{a}_j)=\exp(\mathrm{i}\alpha_{j,q}),$$

whereas

$$m(\mathbf{a}_j,\mathbf{a}_q)=\exp(-\mathrm{i}\alpha_{j,q}/2), \qquad m(\mathbf{a}_q,\mathbf{a}_j)=\exp(\mathrm{i}\alpha_{j,q}/2),$$

so the first system leads to *a*-symmetric and the second one — to *anti*-symmetric factors. It is obvious that the second system leads to operators τ, introduced by Zak, and projective representations, considered by Brown, with the corresponding factors. It means that using the other (equivalent) factor system (e.g. (67)) the appropriate formulae (12) and (23) should be changed — it can be done choosing the other, asymmetric, gauge, for example the Landau gauge (cf. [4, 14, 32]).

This work ends with a general formula for a factor system in MTG. Obviously, the further considerations should lead us toward physics, especially by investigations of the irreducible representations and the periodic boundary conditions.

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Appendix: Relations between letters $A_j^{(\mathbf{k})}$ and $r_j^{(\mathbf{k})}$

Since in the following considerations only one element of n-tuple $\mathbf{k}$ is relevant, namely k_j for a given j, therefore symbols $A_j^{(\mathbf{k})}$ and $r_j^{(\mathbf{k})}$ will be in this section replaced by $A_j^{(k_j)}$ and $r_j^{(k_j)}$, respectively. It means that in the presented formula other entries of $\mathbf{k}$ are fixed and irrelevant.

To check the relation (48) one has to consider separately cases $k_j<-1$, $k_j=-1$, $k_j=0$, and $k_j>0$. In the special cases, $k_j=-1,0$ one obtains

$$\begin{aligned}
\bar{r}_j^{(-1)}\, r_j^{(0)} &= A_j^{(-1)}, \\
\bar{r}_j^{(0)}\, r_j^{(1)} &= A_j^{(0)},
\end{aligned}$$

since $r_j^{(0)}=1_F$. For $k_j>0$ and $k_j<-1$ the following expressions have to be written

$$\begin{aligned}
\bar{r}_j^{(k_j)} r_j^{(k_j+1)} &= \Big(\prod_{l=0}^{k_j-1} A_j^{(l)}\Big)^{-1} \prod_{l=0}^{k_j} A_j^{(l)} = \prod_{l=k_j-1}^{0} \bar{A}_j^{(l)} \prod_{l=0}^{k_j} A_j^{(l)} = A_j^{(k_j)}. \\
\bar{r}_j^{(k_j)} r_j^{(k_j+1)} &= \prod_{l=k_j}^{-1} A_j^{(l)} \prod_{l=-1}^{k_j+1} \bar{A}_j^{(l)} = A_j^{(k_j)}.
\end{aligned}$$

Verifying relation (49) one has to distinguish carefully the cases of infinite and finite translation groups due to differences between definitions (43) and (44). For $k_j \neq N_j-1$ both cases are identical and the definitions of $A_j^{(\mathbf{k})}$ and $r_j^{(\mathbf{k})}$ give (for $k_j>0$)

$$\begin{aligned}
r_j^{(k_j)} &= \Big(x_1^{k_1}\ldots x_j^0\ldots x_n^{k_n} x_j \bar{x}_n^{k_n}\ldots \bar{x}_j^1\ldots \bar{x}_1^{k_1}\Big)\Big(x_1^{k_1}\ldots x_j^1\ldots x_n^{k_n} x_j \bar{x}_n^{k_n}\ldots \bar{x}_j^2\ldots \bar{x}_1^{k_1}\Big) \\
&\quad \times\ldots\Big(x_1^{k_1}\ldots x_j^{k_j-1}\ldots x_n^{k_n} x_j \bar{x}_n^{k_n}\ldots \bar{x}_j^{k_j}\ldots \bar{x}_1^{k_1}\Big) \\
&= x_1^{k_1}\ldots x_j^0\ldots x_n^{k_n} x_j^{k_j} \bar{x}_n^{k_n}\ldots \bar{x}_j^{k_j}\ldots \bar{x}_1^{k_1}.
\end{aligned}$$

The relation (49) for $k_j=0$ is evident and for $k_j<0$ a proof is analogous. It is easy to notice that for a finite translation group (i.e. for $\beta(x_j^{N_j})=1_F$), when the definition (44) has to be used, one obtains

$$r_j^{(N_j)} = x_1^{k_1}\ldots x_j^0\ldots x_n^{k_n} x_j^{N_j} \bar{x}_n^{k_n}\ldots \bar{x}_j^0\ldots \bar{x}_1^{k_1}.$$

Please note, that letters $r_j^{(\mathbf{k})}$ can be considered as an 'ordering' operators (or 'cyclic' generalization of commutator) since

$$x_1^{k_1}\ldots x_j^{k_j}\ldots x_n^{k_n} = \bar{r}_j^{(\mathbf{k})}\Big(x_1^{k_1}\ldots x_j^0\ldots x_n^{k_n} x_j^{k_j}\Big) \tag{A.1}$$

and this formula confirms correspondence between the cyclic permutations $(j\, j{+}1\, \ldots\, n)$ and letters $r_j^{(\mathbf{k})}$ (so as well letters $A_j^{(\mathbf{k})}$).

The relation (48) suggests calculation of products $\bar{r}_j^{(k_j)} r_j^{(k_j+l)}$. Using the identity (49) it is easy to prove that

$$\bar{r}_j^{(k_j)} r_j^{(k_j+l)} = x_1^{k_1}\ldots x_j^{k_j}\ldots x_n^{k_n} x_j^l \bar{x}_n^{k_n}\ldots \bar{x}_j^{k_j+l}\ldots \bar{x}_1^{k_1} = A_{j,l}^{(k_j)},$$

where a symbol $A_{j,l}^{(k_j)}$ is generalization of $A_j^{(k_j)}=A_{j,1}^{(k_j)}$.

The above presented considerations confirms that different alphabets (with the same number of letters) may be used to write down words in a given free group. However, the choice of the alphabet Y is an essential part of the Mac Lane method and is determined by the choice of generators (the set A) and the representatives (the Schreier set S). For example, one may add to generators all (non-zero) multiplicities $k_j\mathbf{a}_j$. According to the Nielsen–Schreier formula (45) this will lead to increase in number of letters in the alphabet Y. However, it seems that letters $r_j^{(\mathbf{k})}$ cannot be obtained by changes in A and/or S sets. On the other hand, these letters have been generalized to all permutations in Eq. (50). It is well-known that cyclic permuations generate the symmetric group S_n. Despite this fact letters $r_\sigma^{(\mathbf{k})}$ are not expressed by letters $r_j^{(\mathbf{k})}$ in an analoguos way. If it were than one could have used only two type of letters, namely $r_{(1\,2\,\ldots\,n)}^{(\mathbf{k})}$ and $r_{(1\,2)}^{(\mathbf{k})}$. For example, taking $n=3$ and $\sigma=(132)$ we have

$$r_{(132)}^{(\mathbf{k})} = x_3^{k_3} x_1^{k_1} x_2^{k_2} \bar{x}_3^{k_3} \bar{x}_2^{k_2} \bar{x}_1^{k_1}.$$

This element of the kernel R can be rewritten using letters $r_j^{(\mathbf{k})}$ in two ways. The first is indirect and is based on the 'translation' (46) and the relation (48). The second method leads directly to letters $r_j^{(\mathbf{k})}$. Going from left to rigth one can add letters x_j^0 in such a way that a permutation determined by full cycle $(j\, j{+}1\, \ldots\, n)$ is obtained. Next the relation (A.1) is apllied and the procedure goes on (after removing unnecessary x_j^0). In the presented example one obtains

$$r_{(132)}^{(\mathbf{k})} = x_3^{k_3} x_1^{k_1} x_2^{k_2} \bar{x}_3^{k_3} \bar{x}_2^{k_2} \bar{x}_1^{k_1}$$

$$
\begin{aligned}
&= (x_2^0 x_3^{k_3} x_1^{k_1}) x_2^{k_2} \bar{x}_3^{k_3} \bar{x}_2^{k_2} \bar{x}_1^{k_1} \\
&= r_1^{(k_1\,0\,k_3)} (x_1^{k_1} x_3^{k_3} x_2^{k_2}) \bar{x}_3^{k_3} \bar{x}_2^{k_2} \bar{x}_1^{k_1} \\
&= r_1^{(k_1,0,k_3)} r_2^{(k_1,k_2,k_3)} x_1^{k_1} x_2^{k_2} x_3^{k_3} \bar{x}_3^{k_3} \bar{x}_2^{k_2} \bar{x}_1^{k_1}.
\end{aligned}
$$

Finally we have obtained

$$
r_{(132)}^{(\mathbf{k})} = r_{(123)}^{(k_1,0,k_3)} r_{(23)}^{(k_1,k_2,k_3)}
$$

in spite of fact that (132) = (123)(123).

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